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Quantum theory of multiphoton lasers II. Systems without detailed balance

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Abstract. Investigations into possible multiphoton laser systems are extended to a less restrictive class than considered in the previous paper. The loss mechanism for the light field is now taken to be a single-photon loss. These systems no longer preserve the property of detailed balance. However, approximate solutions to the laser Fokker–Planck equation may be obtained using a method recently developed by Haken. Explicit solutions are given for the stationary photon distribution for the two-photon laser. This exhibits similar photon statistics to the usual one-photon laser, with no significant increase in relative fluctuations at threshold. The results are equally applicable to multiboson lasers involving say the simultaneous stimulated emission of a photon and a phonon.

1. Introduction

The aim of this paper and the preceding one is to give a quantum-mechanical description of multiphoton lasers, that is, lasers in which the atomic lasing transition involves the emission of more than one photon. In the preceding paper (McNeil and Walls 1975) we examined a special class of such lasers for which detailed balance was satisfied. The loss mechanism for the light field in these lasers was multiphoton absorption of the same order as the stimulated multiphoton emission generated by the pumped atoms. This rather restrictive loss mechanism was introduced in order to preserve detailed balance.

In this paper we consider a less restrictive class of multiphoton lasers which have a single-photon loss mechanism. Obviously detailed balance does not hold for such lasers, and this makes analytic solutions of the laser equations more difficult. However, we may obtain reasonable approximate solutions to the Fokker–Planck equation describing the behaviour of the multiphoton laser using a perturbation technique recently developed by Haken (1973a). This method has been applied so far to interacting chaotic boson fields (Haken 1973b) and to the usual one-photon laser (Haken and Wohrstein 1973, Haken 1973c).

We illustrate how this method is applicable to the general case of the multiphoton laser and obtain explicit solutions for the two-photon laser. We consider both the single-mode and two-mode cases and compare the results with the corresponding results for the usual one-photon laser. The results obtained are equally applicable to multiboson lasers involving the simultaneous stimulated emission of a photon and a phonon.

2. Model and analytical approach

We adopt the same model for the multiphoton laser as described in paper I, except the loss mechanism for the light field is now a single-photon loss. Thus, in place of equation (I2.5) the Hamiltonian describing the photon loss assumes the form

$$H_{\mathrm{FR_F}} = b\Gamma_{\mathrm{F}}^{\dagger}+b^{\dagger}\Gamma_{\mathrm{F}}. \tag{2.1}$$

The density operator for the coupled atom field system obeys the usual equation of motion in the interaction picture

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = \frac{\mathrm{i}}{\hbar}[H_{\mathrm{int}}, \rho]+\Lambda\rho \tag{2.2}$$

where the operator Λ describes the irreversible behaviour incorporating the loss and pumping mechanisms.

The quantum-mechanical operator equation (2.1) may be converted to a classical Fokker–Planck equation for the distribution function $f(u_j, u_j^*, v, v^*, D)$ using standard methods (Haken 1970). The distribution function is related to the density operator via the transformation

$$f = \mathcal{N}\int \exp\left[-\mathrm{i}\left(\sum_j \beta_j^* u_j^* + \sum_j \beta_j u_j + \xi^* v^* + \xi v + \tfrac{1}{2}\zeta D\right)\right] \mathrm{Tr}(O\rho)\prod_{j=1}^{n}\mathrm{d}^2\beta_j \tag{2.3}$$

$$O = \prod_{j=1}^{n}\mathrm{e}^{\mathrm{i}\beta_j^* b_j^\dagger}\prod_{j=1}^{n}\mathrm{e}^{\mathrm{i}\beta_j b_j}\,\mathrm{e}^{\mathrm{i}\xi^* S^+}\,\mathrm{e}^{\mathrm{i}\xi S^-}\,\mathrm{e}^{\mathrm{i}\zeta S_z}. \tag{2.4}$$

The classical variables u_j, v, D and their complex conjugates correspond to the quantum-mechanical operators as follows:

$$\begin{aligned} &u_j \leftrightarrow b_j; && u_j^* \leftrightarrow b_j^\dagger \\ &v \leftrightarrow S^-; && v^* \leftrightarrow S^+ \\ &D \leftrightarrow 2S_z. && \end{aligned} \tag{2.5}$$

u_j, u_j^* represent the amplitude of the field mode j; v, v^* represent the atomic dipole moment, and D represents the atomic level population inversion.

The resulting Fokker–Planck equation may be written, in the steady state, as

$$Lf = (L_0+L_1+L_2)f = 0. \tag{2.6}$$

The operator L has been split up as follows:

$$\begin{aligned} L_0 = -\mathrm{i}g\bigg(& vu_2u_3\ldots u_n\frac{\partial}{\partial u_1}+vu_1u_3\ldots u_n\frac{\partial}{\partial u_2}+\ldots+vu_1\ldots u_{n-1}\frac{\partial}{\partial u_n} \\ & -Du_1u_2\ldots u_n\frac{\partial}{\partial v}+2v^*u_1\ldots u_n\frac{\partial}{\partial D}-\mathrm{cc}+\text{higher order derivatives in } u\text{'s}\bigg) \end{aligned} \tag{2.7}$$

$$L_1 = \frac{1}{2}\eta\left(\frac{\partial}{\partial v}v+\frac{\partial}{\partial v^*}v^*\right)+\frac{1}{2}N\eta\frac{\partial^2}{\partial v\partial v^*} \tag{2.8}$$

$$L_2 = \sum_{j=1}^{n}\left[\kappa_j\left(\frac{\partial}{\partial u_j}u_j+\frac{\partial}{\partial u_j^*}u_j^*\right)+2\kappa_j\bar{n}_j\frac{\partial^2}{\partial u_j\partial u_j^*}\right]+\frac{1}{2}\gamma_{\parallel}\left(\frac{\partial}{\partial v}v+\frac{\partial}{\partial v^*}v^*\right)+N\omega_{12}\frac{\partial^2}{\partial v\partial v^*}$$
$$+\gamma_{\parallel}\frac{\partial}{\partial D}(D-D_0)+\gamma_{\parallel}N\frac{\partial^2}{\partial D^2}. \tag{2.9}$$

L_0 represents the basic atom–field interaction. L_1 and L_2 represent the effects of the reservoirs. N is the number of atoms.

η is the atomic phase decay constant. $\gamma_{\parallel}$ is the longitudinal relaxation time, with

$$\gamma_{\parallel} = \omega_{12}+\omega_{21}. \tag{2.10}$$

D_0 is the atomic inversion due to pumping and incoherent relaxation processes from the atom–reservoir coupling alone, and

$$D_0 = \frac{\omega_{12}-\omega_{21}}{\omega_{12}+\omega_{21}}; \tag{2.11}$$

ω_{12} and ω_{21} are the reservoir-induced transition rates between atomic levels 1 and 2. κ_j is the cavity damping constant for the jth photon mode, and $\bar{n}_j$ is the mean photon number in the thermal reservoir for the jth mode.

3. Solution of the Fokker–Planck equation

To solve equation (2.6) we adopt the general method described in detail by Haken (1973a). First we must seek the fundamental solutions, or constants of motion h_j, of the unperturbed equation

$$L_0 f = 0. \tag{3.1}$$

We then require that L be such that any function of the constants h_j is again a solution of (3.1).

We obtain the following constants of motion for L:

$$h_0 = \prod_j u_j v^* + \prod_j u_j^* v \tag{3.2}$$

$$h_1 = |v|^2+\tfrac{1}{4}D^2 \tag{3.3}$$

$$h_2 = |u_1|^2+\tfrac{1}{2}D \tag{3.4}$$

$$h_{j+1} = |u_1|^2-|u_j|^2, \qquad j = 2, \ldots. \tag{3.5}$$

The perturbation technique assumes $\kappa_j, \gamma_{\parallel}, \eta$ are much smaller than g, and involves the solution of (2.6) in the subspace of variables spanned by the h_j. Following Haken and Worhstein (1973) we assume $\eta \gg \kappa_j, \gamma_{\parallel}$, so that a further perturbation may be taken on $L_1 f = 0$.

If we assume a strong phase damping, the terms in h_0 may be dropped. This assumption also ensures that L_0 has the desired property that any function of the h_j is again a solution of (3.1).

In this case, the equation $L_1 f_0 = 0$ is then

$$L_1 f_0(h_1, h_2, \ldots) = 0 \tag{3.6}$$

which has the solution (Haken and Worhstein 1973)

$$f_0 = e^{-2h_1/N} f_1(h_2, h_3, \ldots) \tag{3.7}$$

where f_1 is to be determined according to the perturbation method. The Fokker–Planck equation (2.6) is converted to one in the subspace spanned by the h_j, giving

$$\left(\sum_j \frac{\partial}{\partial h_j} G^{(j)} + \sum_i \sum_j \frac{\partial}{\partial h_i} G^{(i,j)} \frac{\partial}{\partial h_j}\right) f_0(h_j) = 0 \tag{3.8}$$

where the G's are as prescribed in Haken (1973a).

To eliminate h_1, we insert the form (3.7) in (3.8), and integrate over h_1. This yields

$$\left(\sum_{j\neq 1} \frac{\partial}{\partial h_j} (\hat{G}^{(j)} + \hat{G}^{(j,1)}) + \sum_{i\neq 1} \sum_{j\neq 1} \frac{\partial}{\partial h_i} \hat{G}^{(i,j)} \frac{\partial}{\partial h_j}\right) f_1(h_2, \ldots) = 0 \tag{3.9}$$

where

$$\hat{G}^{(\alpha,\beta)} = \int_0^\infty dh_1 G^{(\alpha,\beta)} \exp(-2h_1/N), \qquad \beta \neq 1$$

and

$$\hat{G}^{(j,1)} = \int_0^\infty dh_1 G^{(j,1)} \frac{\partial}{\partial h_1} \exp(-2h_1/N). \tag{3.10}$$

(3.9) is our final equation, which is to be solved for the desired system.

4. Application to two-photon lasers

4.1. Two modes

Here we take the atomic de-excitation to involve the emission of two photons, into distinct modes. The interaction Hamiltonian is then

$$H_{\text{int}} = \hbar g(S^- b_1^\dagger b_2^\dagger + S^+ b_1 b_2) \tag{4.1}$$

with the corresponding Fokker–Planck operator

$$L_0 = -\mathrm{i}g\left(vu_2^* \frac{\partial}{\partial u_1} + vu_1^* \frac{\partial}{\partial u_2} - Du_1u_2 \frac{\partial}{\partial v} - 2u_1u_2v^* \frac{\partial}{\partial D}\right.$$
$$\left. - \text{cc} + \text{higher order derivatives in } u_1, u_2\right). \tag{4.2}$$

The constant of motion besides h_1 and h_2 is

$$h_3 = |u_1|^2 - |u_2|^2. \tag{4.3}$$

The Fokker–Planck equation (3.9) is:

$$\left[\frac{\partial}{\partial h_2}(\hat{G}^{(2)} + \hat{G}^{(2,1)}) + \frac{\partial}{\partial h_3}(\hat{G}^{(3)} + \hat{G}^{(3,1)}) + \frac{\partial}{\partial h_2}\hat{G}^{(2,3)}\frac{\partial}{\partial h_3} + \frac{\partial}{\partial h_3}\hat{G}^{(3,2)}\frac{\partial}{\partial h_2}\right.$$
$$\left. + \frac{\partial}{\partial h_2}\hat{G}^{(2,2)}\frac{\partial}{\partial h_2} + \frac{\partial}{\partial h_3}\hat{G}^{(3,3)}\frac{\partial}{\partial h_3}\right] f_1(h_2, h_3) = 0 \tag{4.4}$$

with

$$\hat{G}^{(\alpha)} = \int_0^\infty dh_1 \int d^2u_1\, d^2u_2\, d^2v\, dD\, \delta(h_1 - h_1(v, D))\, \delta(h_2 - h_2(u_1, D))$$
$$\times \delta(h_3 - h_3(u_1, u_2))\Omega^{(\alpha)} \exp(-2h_1/N) \tag{4.5}$$
$$\Omega^{(2)} = 2\kappa_1|u_1|^2 + \tfrac{1}{2}\gamma_{\|}(D - D_0) \tag{4.6}$$
$$\Omega^{(3)} = 2\kappa_1|u_1|^2 - 2\kappa_2|u_2|^2 \tag{4.7}$$
$$\Omega^{(2,1)} = \frac{1}{4}\gamma_{\|} ND \frac{\partial}{\partial h_1} \tag{4.8}$$
$$\Omega^{(3,1)} = 0 \tag{4.9}$$
$$\Omega^{(2,3)} = \Omega^{(3,2)} = 2\kappa_1\bar{n}_1|u_1|^2 \tag{4.10}$$
$$\Omega^{(2,2)} = 2\kappa_1\bar{n}_1|u_1|^2 + \tfrac{1}{4}\gamma_{\|}N \tag{4.11}$$
$$\Omega^{(3,3)} = 2\kappa_1\bar{n}_1|u_1|^2 + 2\kappa_2\bar{n}_2|u_2|^2. \tag{4.12}$$

Evaluation of the non-zero integrals gives:

$$\hat{G}^{(2)} = \pi^3 N\{\tfrac{1}{4}(2\kappa_1 - \gamma_{\|})N \exp[-2(h_2 - m)^2/N] + (2\kappa_1 h_2 - \tfrac{1}{2}D_0\gamma_{\|})I(h_2, h_3)\} \tag{4.13}$$
$$\hat{G}^{(3)} = \pi^3 N\{\tfrac{1}{2}(\kappa_1 - \kappa_2)N \exp[-2(h_2 - m)^2/N] + 2[(\kappa_1 - \kappa_2)h_2 + \kappa_2 h_3]I(h_2, h_3)\} \tag{4.14}$$
$$\hat{G}^{(2,1)} = \tfrac{1}{4}\pi^3 N^2 \gamma_{\|} \exp[-2(h_2 - m)^2/N] \tag{4.15}$$
$$\hat{G}^{(2,3)} = \hat{G}^{(3,2)} = \pi^3 N\{\tfrac{1}{2}\kappa_1\bar{n}_1 N \exp[-2(h_2 - m)^2/N] + 2\kappa_1\bar{n}_1 h_2 I(h_2, h_3)\} \tag{4.16}$$
$$\hat{G}^{(2,2)} = \pi^3 N\{\tfrac{1}{2}\kappa_1\bar{n}_1 N \exp[-2(h_2 - m)^2/N] + (2\kappa_1\bar{n}_1 h_2 + \tfrac{1}{4}\gamma_{\|}N)I(h_2, h_3)\} \tag{4.17}$$
$$\hat{G}^{(3,3)} = \pi^3 N\{\tfrac{1}{2}(\kappa_1\bar{n}_1 + \kappa_2\bar{n}_2)N \exp[-2(h_2 - m)^2/N]$$
$$+ 2[(\kappa_1\bar{n}_1 + \kappa_2\bar{n}_2)h_2 - \kappa_2\bar{n}_2 h_3]I(h_2, h_3)\} \tag{4.18}$$

where

$$\left.\begin{aligned} m &= \max(0, h_3) \\ I(h_2, h_3) &= \int_{-\infty}^{(h_2 - m)} e^{-2x^2/N}\, dx. \end{aligned}\right\} \tag{4.19}$$

In most applications $\bar{n}$ is vanishingly small, so we may set $\bar{n}_1, \bar{n}_2 \simeq 0$. Hence (4.4) reduces to

$$\left(\frac{\partial}{\partial h_2}\left\{\frac{1}{2}\kappa_1 N \exp[-2(h_2 - m)^2/N] + (2\kappa_1 h_2 - \tfrac{1}{2}D_0\gamma_{\|})I(h_2, h_3)\right\}\right.$$
$$+ \frac{\partial}{\partial h_3}\left\{\frac{1}{2}(\kappa_1 - \kappa_2)N \exp[-2(h_2 - m)^2/N] + 2[(\kappa_1 - \kappa_2)h_2 + \kappa_2 h_3]I(h_2, h_3)\right\}$$
$$\left. + \frac{\partial}{\partial h_2}\left(\frac{1}{4}\gamma_{\|} N I(h_2, h_3)\right)\frac{\partial}{\partial h_2}\right) f_1(h_2, h_3) = 0. \tag{4.20}$$

We anticipate $h_2 = n_1 + \tfrac{1}{2}D$ and $h_2 - h_3 = n_2 + \tfrac{1}{2}D \sim N^{1/2}$ around onset of lasing, so that for large N, we may set the upper integration limit in $I(h_2, h_3)$ to be ∞, so that $I(h_2, h_3)$ becomes $(\tfrac{1}{2}\pi N)^{1/2}$. Further, the terms $\kappa_j \exp[-2(h_2 - m)^2/N]$ may be ignored when compared with the other terms in (4.20). Hence (4.20) becomes

$$\left(\frac{\partial}{\partial h_2}\left(2\kappa_1 h_2 - \frac{1}{2}D_0\gamma_{\|}\right) + \frac{\partial}{\partial h_3}2[(\kappa_1-\kappa_2)h_2+\kappa_2 h_3] + \frac{1}{4}\gamma_{\|}N\frac{\partial^2}{\partial h_2^2}\right) f_1(h_2, h_3) = 0 \tag{4.21}$$

which has the solution

$$f_1(h_2, h_3) = \mathcal{N}\exp\left[-2\frac{(\kappa_1+\kappa_2)^2}{N\gamma_{\|}(\kappa_1-\kappa_2)^2}\left(2\kappa_1 h_2^2 + 2\kappa_2(h_2-h_3)^2 - \frac{8\kappa_1\kappa_2}{\kappa_1-\kappa_2}h_2(h_2-h_3) + \frac{D_0\gamma_{\|}(\kappa_1-\kappa_2)}{\kappa_1+\kappa_2}h_2 + \frac{D_0\gamma_{\|}(\kappa_2-\kappa_1)}{\kappa_1+\kappa_2}(h_2-h_3)\right)\right] \tag{4.22}$$

provided $\kappa_1 \neq \kappa_2$.

The distribution for the light field alone is obtained by inserting (4.22) in (3.7) to obtain the full distribution, and then integrating over v and D:

$$f_1(n_1, n_2) = \mathcal{N}\exp\{2d[-q_1 n_1 - q_2 n_2 + b_1(1-\beta_1)n_1^2 + b_2(1-\beta_2)n_2^2 + 8pn_1 n_2]\} \tag{4.23}$$

where:

$$\begin{aligned} d &= (\kappa_1+\kappa_2)^2/(\kappa_1-\kappa_2)^2 \\ q_1 &= -q_2 = (D_0/N)(\kappa_1-\kappa_2)/(\kappa_1+\kappa_2) \\ b_j &= 2\kappa_j/N\gamma_{\|} \\ \beta_j &= b_j/(b+c) \\ b &= 2(\kappa_1+\kappa_2)/N\gamma_{\|} \\ c &= 1/2N. \\ p &= 1/(\kappa_1+\kappa_2) - 1/(N\gamma_{\|}(b+c)). \end{aligned} \tag{4.24}$$

Integration over mode 2 (say) gives the single-mode distribution:

$$f(n_1) = \mathcal{N}\exp[a_1' n_1 - b_1'(1-\beta)n_1^2] \tag{4.25}$$

where

$$\begin{aligned} a_1' &= 2D_0/N(1-\beta_2) \\ b_1' &= 2b_1/(1-\beta_2) \\ \beta &= b/(b+c). \end{aligned} \tag{4.26}$$

Since $\kappa_1 \ll \gamma_{\|}$ in practice, $\beta \ll 1$, and hence (4.25) reduces to the well known result of photon statistics for the usual one-photon laser (Risken 1965). An analogous expression is obtained for the distribution of the other mode. Equation (4.25) shows that in this theory for the two-photon laser, each mode separately exhibits lasing action of the usual type.

If $\kappa_1 = \kappa_2 = \kappa$, say, the solution of (4.21) is

$$f(n_1, n_2) = \delta(n_1 - n_2)\exp[a(1-\beta)n_1 - b(1-\beta)n_1^2] \tag{4.27}$$

where

$$\begin{aligned} a &= 2D_0/N \\ b &= 4\kappa/\gamma_{\|}N \\ c &= 2/N \\ \beta &= b/(b+c). \end{aligned} \tag{4.28}$$

The above results apply equally well to two-boson lasers where for $\kappa_1 \neq \kappa_2$ the two bosons need not necessarily be of the same character. For example, a particular two-boson laser may involve the simultaneous stimulated emission of a photon and a phonon. Experimental descriptions of such lasers have already been given (see Johnson *et al* 1963, 1964, Johnson *et al* 1966, where further references are given).

4.2. Single mode

Here we consider the two photons to be emitted into the same mode. The interaction Hamiltonian is thus

$$H_{\rm int} = \hbar g(S^- b^{\dagger 2} + S^+ b^2). \tag{4.29}$$

The constants of motion of interest are

$$h_1 = |v|^2 + \tfrac{1}{4}D^2 \tag{4.30}$$

$$h_2 = |u|^2 + D. \tag{4.31}$$

The Fokker–Planck equation in the h_2 subspace is

$$\left(\frac{\partial}{\partial h_2}(\hat{G}^{(2)} + \hat{G}^{(2,1)}) + \frac{\partial}{\partial h_2}\hat{G}^{(2,2)}\frac{\partial}{\partial h_2}\right) f_1(h_2) = 0 \tag{4.32}$$

where

$$\hat{G}^{(2)} = \pi^2 N[(2\kappa h_2 - \gamma_{\|} D_0)I(h_2) + (2\kappa - \gamma_{\|})N\,\mathrm{e}^{-h_2^2/2N}] \tag{4.33}$$

$$\hat{G}^{(2,1)} = \pi^2 N^2 \gamma_{\|}\,\mathrm{e}^{-h_2^2/2N} \tag{4.34}$$

$$\hat{G}^{(2,2)} = \pi^2 N[(2\kappa\bar{n}h_2 + \gamma_{\|}N)I(h_2) + 2\kappa\bar{n}\,\mathrm{e}^{-h_2^2/2N}] \tag{4.35}$$

where

$$I(h_2) = \int_{-\infty}^{h_2} \mathrm{e}^{-x^2/2N}\,\mathrm{d}x. \tag{4.36}$$

The solution of (4.32) is

$$f_1(h_2) = \mathcal{N}\exp\left(-\int^{h_2}\mathrm{d}h_2'\,\frac{2\kappa N\,\mathrm{e}^{-h_2'^2/2N} + (2\kappa h_2' - D_0\gamma_{\|})I(h_2')}{2\kappa\bar{n}\,\mathrm{e}^{-h_2'^2/2N} + (2\kappa\bar{n}h_2' + \gamma_{\|}N)I(h_2')}\right). \tag{4.37}$$

Around threshold, the $\kappa\,\mathrm{e}^{-h_2'^2/2N}$ terms may be ignored. Further, $2\kappa\bar{n}h_2' \ll \gamma_{\|}N$, so that near threshold,

$$f_1(h_2) \simeq \mathcal{N}\exp\left(-\frac{\kappa}{\gamma_{\|}N}h_2^2 + \frac{D_0}{N}h_2\right). \tag{4.38}$$

Substituting (4.37) into (3.7) and integrating over D and v give the boson field distribution

$$f(n) = \mathcal{N}\exp[\tilde{a}(1-\tilde{\beta})n - \tilde{b}(1-\tilde{\beta})n^2] \tag{4.39}$$

where

$$\begin{aligned} \tilde{a} &= D_0/N \\ \tilde{b} &= \kappa/\gamma_{\|}N \\ c &= 1/2N \\ \tilde{\beta} &= \tilde{b}/(\tilde{b}+c). \end{aligned} \tag{4.40}$$

For $\tilde{\beta} \ll 1$, (4.39) reduces to the usual laser threshold distribution.

5. Comparison with the one-photon laser

In the limit $\kappa \ll \gamma_{\|} \ll \eta \ll g$, the photon distribution in an ordinary laser is (Haken and Worhstein 1973)

$$f(n) = \mathcal{N} \exp(an - bn^2) \tag{5.1}$$

where

$$\begin{aligned} a &= 2D_0/N \\ b &= 4\kappa/\gamma_{\|}N. \end{aligned} \tag{5.2}$$

b is the non-linear parameter, and a is the threshold parameter, with $a = 0$ (ie $\omega_{21} = \omega_{12}$) giving the threshold.

The mean at threshold for distributions of type (5.1) is

$$\langle n \rangle_{\text{thr}} = \frac{1}{(\pi b)^{1/2}} \tag{5.3}$$

and the threshold variance is

$$\sigma^2_{\text{thr}} = \left(\frac{1}{2} - \frac{1}{\pi}\right)\frac{1}{b} + \frac{1}{(\pi b)^{1/2}}. \tag{5.4}$$

The square of the relative fluctuation at threshold is

$$\frac{\sigma^2_{\text{thr}}}{\langle n \rangle^2_{\text{thr}}} = \frac{1}{(\frac{1}{2}\pi - 1) + (\pi b)^{1/2}}. \tag{5.5}$$

The one-mode distributions for the lasers we have discussed have the same form as (5.1), but with modified parameters a and b, so that the threshold mean, variance and relative fluctuation will take the same form as equations (5.3) to (5.5).

In the case of the two-mode two-photon laser, the single-mode non-linear parameters are the same as for the corresponding one-photon laser (ie one with the same $\gamma_{\|}$ and κ, see equations (4.24) and (4.26)). Thus, for this two-photon laser, $\langle n \rangle_{\text{thr}}$, σ_{thr} and $\sigma_{\text{thr}}/\langle n \rangle_{\text{thr}}$ are the same as for the corresponding one-photon laser.

In the case of the one-mode two-photon laser the non-linear parameter is one quarter that of the corresponding one-photon laser (see equation (4.40)). Thus $\langle n \rangle_{\text{thr}}$ will be increased by a factor of two. σ^2_{thr} will also be altered. Since $b \ll 1$, to a good approximation we may write

$$\sigma^2_{\text{thr}} \simeq \left(\frac{1}{2} - \frac{1}{\pi}\right)\frac{1}{b} \tag{5.6}$$

and

$$\frac{\sigma_{\text{thr}}}{\langle n \rangle_{\text{thr}}} \simeq (\tfrac{1}{2}\pi - 1)^{1/2}. \tag{5.7}$$

According to equation (5.6), the width σ_{thr} is increased by a factor of approximately two. This result is not unexpected, since recent work on two-photon emission (Lambropolous 1967, McNeil and Walls 1974) shows that the two-photon emission process is noisier than the one-photon emission process. However, we observe that, under the approximation (5.7), the relative fluctuation is the same as that for the corresponding one-photon laser. Thus the effect of damping is to inhibit the noise somewhat, and allow the production of two-photon laser light with no significant increase in relative noise over the one-photon laser.

6. Conclusion

We have shown that it is theoretically possible to obtain a photon field distribution for a multiphoton laser which has qualitatively the same form as the usual laser threshold distribution. This distribution function exhibits a close analogy with the distribution function describing the threshold region of a phase transition (Graham and Haken 1970, De Giorgio and Scully 1970, Grossman and Ritcher 1971). Hence we may say that the separate field modes exhibit second phase transitions far from equilibrium in a like manner to the one-photon laser. In the case considered in this paper the order parameter is again the number of photons in the mode.

In the two-mode two-photon laser we find no significant difference in the threshold means and widths compared with the one-photon laser, implying that the modes behave separately rather like the usual one-photon lasing modes. The single-mode two-photon laser exhibits an increase in variance and mean by a factor of approximately two at threshold, although the relative fluctuation $\sigma_{\text{thr}}/\langle n\rangle_{\text{thr}}$ shows no significant difference from that for the one-photon laser. Non-thermal light distributions may also be achieved for higher-order multiphoton lasers with a corresponding increase in the variance and mean of the single-mode distribution.

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